ON THE REDUCTION OF THE EQUATIONS OF MOTION OF A GYROHORIZONCOMPASS AND TWO-GYROSCOPE VERTICAL

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Some results of [1] are generalized. The use of simplified equations of Geckeler for arbitrary motion of a gyrohorizoncompass suspension point on the earth sphere is justified with the aid of a theorem by Erugin [2]. An analogous question applicable to a two-gyroscope vertical is also considered [3].

1. In the absence of damping the equations of perturbed motion of a gyrohorizoncompass are [4]

$$ml v \frac{d\alpha}{dt} + ml \frac{dv}{dt}\alpha - mgl\beta - \Omega 2B \sin \varepsilon^\circ \delta = 0 \qquad \frac{d\beta}{dt} + \frac{v}{R}\alpha - \Omega\gamma = 0 \tag{1.1}$$

$$\frac{d\gamma}{dt} + \frac{2B\sin\varepsilon^\circ}{mlR}\delta + \Omega\beta = 0, \quad \frac{d}{dt}(2B\sin\varepsilon^\circ\delta) - mgl\gamma + mlv\Omega\alpha = 0$$

$$v = \sqrt{(RU\cos\varphi + v_E)^2 + v_N^2}, \; \Omega = u\sin\varphi + \frac{v_E}{R}\tan\varphi - \dot{\alpha}^*, \quad \alpha^* = -\frac{v_N}{Ru\cos\varphi + v_E}$$

Here B is the kinetic moment of the gyroscope rotor; ε° is some equilibrium position for the separation angle of the gyroscopes; α is the deviation angle for the gyrosphere axis in azimuth; β is the lift angle for the north end of the gyrosphere axis above the surface tangent to the earth sphere at the point of gyrosphere suspension; γ is the gyrosphere rotation angle about the line north-south; δ is the gyroscopes' rotation angle about their frames, defining the perturbed location of gyro rotor axes relative to the gyrosphere; $m = P/g$ is the mass of the gyrosphere (P is the weight of the gyrosphere, g the acceleration of gravity); l is the metacentric height of the gyrosphere; R is the earth radius; U is the earth angular velocity; φ is ship's location latitude (geocentric); v_E, v_N are east and north velocity components, respectively, of the gyrosphere suspension point relative to earth surface.

Let the ship perform maneuvers on a given latitude φ.

We will introduce new variables x_j (j = 1, 2, 3, 4) by formulas

$$\alpha = \frac{RU\cos\varphi}{v}x_1, \qquad \beta = x_2, \qquad \gamma = x_3, \qquad \delta = \frac{\sin\varphi}{\sin\varepsilon^\circ}x_4 \tag{1.2}$$

The system (1.1) for new variables becomes [1]

$$\frac{dx_1}{dt} - \frac{\nu^2}{U\cos\varphi}x_2 - \lambda_1\tan\varphi\Omega x_4 = 0, \qquad \frac{dx_2}{dt} + U\cos\varphi x_1 - \Omega x_3 = 0$$
$$\frac{dx_3}{dt} + \frac{2B\sin\varphi\nu^2}{Pl}x_4 + \Omega x_2 = 0, \qquad \frac{dx_4}{dt} - \frac{Pl}{2B\sin\varphi}x_3 + \frac{1}{\lambda_1}\cot\varphi\Omega x_1 = 0 \tag{1.3}$$
$$\left(\nu = \sqrt{\frac{g}{R}}, \ \lambda_1 = \frac{2B\nu^2}{PlU}\right)$$

Koshliakov [1] suggested the substitution

$$\xi_1 = x_1\cos\theta - \frac{\nu}{U\cos\varphi}x_2\cos\theta + \frac{\nu}{U\cos\varphi}x_3\sin\theta - \lambda_1\tan\varphi x_4\sin\theta \tag{1.4}$$
$$\xi_2 = \frac{U\cos\varphi}{\nu}x_1\cos\theta + x_2\cos\theta - x_3\sin\theta - \frac{\nu 2B\sin\varphi}{Pl}x_4\sin\theta \qquad \left(\theta(t) = \int_0^t \Omega(\tau)\,d\tau\right)$$
$$\xi_3 = \frac{U\cos\varphi}{\nu}x_1\sin\theta + x_2\sin\theta + x_3\cos\theta + \frac{\nu 2B\sin\varphi}{Pl}x_4\cos\theta$$
$$\xi_4 = \frac{1}{\lambda_1}\cot\varphi x_1\sin\theta - \frac{Pl}{\nu 2B\sin\varphi}x_2\sin\theta - \frac{Pl}{\nu 2B\sin\varphi}x_3\cos\theta + x_4\cos\theta$$

This substitution reduces the system (1.3) to the Schuler-Geckeler system

$$\frac{d\xi_1}{dt} - \frac{\nu^2}{U\cos\varphi}\xi_2 = 0, \qquad \frac{d\xi_3}{dt} + \frac{2B\sin\varphi\nu^2}{Pl}\xi_4 = 0$$
$$\frac{d\xi_2}{dt} + U\cos\varphi\,\xi_1 = 0, \qquad \frac{d\xi_4}{dt} - \frac{Pl}{2B\sin\varphi}\xi_3 = 0\,. \tag{1.5}$$

2. Substitution (1.4) is applicable for any form of the function $\Omega(t)$. Nevertheless, [1] presents its substantiation only for a special case when $\Omega(t)$ is a periodic function of time. Equations (1.5), however, can be justified for any form of the function $\Omega(t)$ by means of a theorem due to Erugin [2].

The system (1.3) is solved in a similar way to that suggested in [4]. Let us represent the system (1.3) in the form

$$\frac{d}{dt}\left(\frac{U\cos\varphi}{\nu}x_1\right) - \nu x_2 - \frac{2B\sin\varphi\nu}{Pl}\Omega x_4 = 0, \qquad \frac{dx_2}{dt} + U\cos\varphi x_1 - \Omega x_3 = 0 \tag{2.1}$$

$$\frac{dx_3}{dt} + \frac{2B\sin\varphi\nu^2}{Pl}x_4 + \Omega x_2 = 0,\ \frac{d}{dt}\left(\frac{2B\sin\varphi\nu}{Pl}x_4\right) - \nu x_3 + \frac{U\cos\varphi}{\nu}\Omega x_1 = 0$$

Introducing two complex-valued functions of t by

$$\chi(t) = \frac{U\cos\varphi}{\nu}x_1 + ix_2, \qquad \mu(t) = x_3 - i\,\frac{2B\sin\varphi\nu}{Pl}x_4\ (i = \sqrt{-1}) \tag{2.2}$$

The system (2.1) is reduced to two equations of the form

$$\frac{d\chi}{dt} + i\nu\chi = i\Omega\mu, \qquad \frac{d\mu}{dt} + i\nu\mu = i\Omega\chi \tag{2.3}$$

which yield the following equations

$$\frac{d}{dt}(\chi+\mu) + i(\nu-\Omega)(\chi+\mu) = 0, \qquad \frac{d}{dt}(\chi-\mu) + i(\nu+\Omega)(\chi-\mu) = 0 \tag{2.4}$$

These are easily integrated. We have

$$\chi+\mu = C_1\exp\left(-i\int_0^t(\nu-\Omega)dt\right), \qquad \chi-\mu = C_2\exp\left(-i\int_0^t(\nu+\Omega)\,dt\right) \tag{2.5}$$

Here C_1, C_2 are arbitrary constants. The general solution of system (2.3) is of the form

$$\chi = \frac{1}{2}e^{-i\nu t}(C_1e^{i\theta} + C_2e^{-i\theta}), \qquad \mu = \frac{1}{2}e^{-i\nu t}(C_1e^{i\theta} - C_2e^{-i\theta}) \tag{2.6}$$

3. It follows from the solution (2.6) that the integral matrix of the system (2.3) has the structure

$$P = e^{-i\nu t}Z, \qquad Z(t) = \frac{1}{2}\begin{Vmatrix} e^{i\theta} & e^{-i\theta} \\ e^{i\theta} & -e^{-i\theta} \end{Vmatrix} \quad \text{— Liapunov type matrix} \tag{3.1}$$

It follows, therefore, that on the basis of a theorem due to Erugin [2], the system (2.3) is reducible for any form of the function $\Omega(t)$. The substitution

$$Y = Z^{-1}X, \qquad Y(t) = \begin{Vmatrix} y_1 \\ y_2 \end{Vmatrix}, \quad Z^{-1}(t) = \begin{Vmatrix} e^{-i\theta} & e^{-i\theta} \\ e^{i\theta} & -e^{i\theta} \end{Vmatrix}, \quad X(t) = \begin{Vmatrix} \chi \\ \mu \end{Vmatrix} \tag{3.2}$$

transforms the system (2.3) into the system with constant coefficients

$$\frac{dy_1}{dt} + i\nu y_1 = 0, \qquad \frac{dy_2}{dt} + i\nu y_2 = 0 \tag{3.3}$$

Inverse transformation from the variables y_1, y_2 to the variables χ, μ according to (3.2) is of the form

$$X = ZY \tag{3.4}$$

Letting

$$y_1 = \eta_1 + i\eta_2, \qquad y_2 = \eta_3 + i\eta_4 \tag{3.5}$$

we have on the basis of (3.3) the Schuler-Geckeler system for the variables η_j

$$\frac{d\eta_1}{dt} - \nu\eta_2 = 0, \quad \frac{d\eta_2}{dt} + \nu\eta_1 = 0, \quad \frac{d\eta_3}{dt} - \nu\eta_4 = 0, \quad \frac{d\eta_4}{dt} + \nu\eta_3 = 0 \tag{3.6}$$

From (3.2) and considering (2.2) and (3.5), we obtain the formulas for a non-singular transformation from the variables x_j to the variables η_j of the form

$$\begin{aligned}
\eta_1 &= \frac{U\cos\varphi}{\nu} x_1\cos\theta + x_2\sin\theta + x_3\cos\theta - \frac{2B\sin\varphi\nu}{Pl} x_4\sin\theta \\
\eta_2 &= -\frac{U\cos\varphi}{\nu} x_1\sin\theta + x_2\cos\theta - x_3\sin\theta - \frac{2B\sin\varphi\nu}{Pl} x_4\cos\theta \\
\eta_3 &= \frac{U\cos\varphi}{\nu} x_1\cos\theta - x_2\sin\theta - x_3\cos\theta - \frac{2B\sin\varphi\nu}{Pl} x_4\sin\theta \\
\eta_4 &= \frac{U\cos\varphi}{\nu} x_1\sin\theta + x_2\cos\theta - x_3\sin\theta + \frac{2B\sin\varphi\nu}{Pl} x_4\cos\theta
\end{aligned} \tag{3.7}$$

The formulas for inverse transformation from the variables η_j to the variables x_j are

$$\begin{aligned}
x_1 &= \frac{1}{2}\frac{\nu}{U\cos\varphi}(\eta_1\cos\theta - \eta_2\sin\theta + \eta_3\cos\theta + \eta_4\sin\theta) \\
x_2 &= \frac{1}{2}(\eta_1\sin\theta + \eta_2\cos\theta - \eta_3\sin\theta + \eta_4\cos\theta) \\
x_3 &= \frac{1}{2}(\eta_1\cos\theta - \eta_2\sin\theta - \eta_3\cos\theta - \eta_4\sin\theta) \\
x_4 &= \frac{1}{2}\frac{Pl}{\nu 2B\sin\varphi}(-\eta_1\sin\theta - \eta_2\cos\theta - \eta_3\sin\theta + \eta_4\cos\theta)
\end{aligned} \tag{3.8}$$

4. The preceding theory is applicable virtually without any alteration to the equations of a two-gyroscope vertical as well, given in [3]:

$$\begin{aligned}
&mav\frac{d\alpha}{dt} + ma\frac{dv}{dt}\alpha - mga\beta + \Omega\, 2B\cos\theta^*\delta = 0 \quad \frac{d\beta}{dt} + \frac{v}{R}\alpha - \Omega\gamma = 0 \\
&\frac{d\gamma}{dt} - \frac{2B\cos\theta^*}{maR}\delta + \Omega\beta = 0, \quad \frac{d}{dt}(2B\cos\theta^*\delta) + mga\gamma - mav\Omega\alpha = 0
\end{aligned} \tag{4.1}$$

Function $\theta^*(t)$ satisfies the condition

$$\theta^*(t) = \sin^{-1}\frac{mav}{2B}$$

The remaining notation in system (4.1) is the same as in (1.1), with a having the same meaning as l. Let us introduce new variables z_j by formulas

$$\alpha = \frac{RU\cos\varphi}{v} z_1, \qquad \beta = z_2, \qquad \gamma = z_3, \qquad \delta = \frac{\cos\varphi}{\cos\theta^*} z_4 \tag{4.2}$$

The system (4.1) will become

$$\frac{dz_1}{dt}-\frac{\nu^2}{U\cos\varphi}z_2+\lambda_2\Omega z_4=0,\quad \frac{dz_3}{dt}-\frac{2B\cos\varphi\nu^2}{Pa}z_4+\Omega z_2=0$$
$$\frac{dz_2}{dt}+U\cos\varphi z_1-\Omega z_3=0,\quad \frac{dz_4}{dt}+\frac{Pa}{2B\cos\varphi}z_3-\frac{1}{\lambda_2}\Omega z_1=0 \qquad \left(\lambda_2=\frac{2B\nu^2}{PaU}\right) \tag{4.3}$$

With analogous reasoning as above, one can show that by means of the non-singular substitution

$$\begin{aligned}
\zeta_1 &= \frac{U\cos\varphi}{\nu}z_1\cos\theta+z_2\sin\theta+z_3\cos\theta+\frac{2B\cos\varphi\nu}{Pa}z_4\sin\theta\\
\zeta_2 &= -\frac{U\cos\varphi}{\nu}z_1\sin\theta+z_2\cos\theta-z_3\sin\theta+\frac{2B\cos\varphi\nu}{Pa}z_4\cos\theta\\
\zeta_3 &= \frac{U\cos\varphi}{\nu}z_1\cos\theta-z_2\sin\theta-z_3\cos\theta+\frac{2B\cos\varphi\nu}{Pa}z_4\sin\theta\\
\zeta_4 &= \frac{U\cos\varphi}{\nu}z_1\sin\theta+z_2\cos\theta-z_3\sin\theta-\frac{2B\cos\varphi\nu}{Pa}z_4\cos\theta
\end{aligned} \tag{4.4}$$

the system (4.3) is reducible for any form of the function $\Omega(t)$ to the Schuler-Geckeler system

$$\frac{d\zeta_1}{dt}-\nu\zeta_2=0,\quad \frac{d\zeta_2}{dt}+\nu\zeta_1=0,\quad \frac{d\zeta_3}{dt}-\nu\zeta_4=0,\quad \frac{d\zeta_4}{dt}+\nu\zeta_3=0 \tag{4.5}$$

The formulas for inverse transformation from the variables ζ_j to the variables z_j are

$$\begin{aligned}
z_1 &= \frac{1}{2}\frac{\nu}{U\cos\varphi}(\zeta_1\cos\theta-\zeta_2\sin\theta+\zeta_3\cos\theta+\zeta_4\sin\theta)\\
z_2 &= \frac{1}{2}(\zeta_1\sin\theta+\zeta_2\cos\theta-\zeta_3\sin\theta+\zeta_4\cos\theta)\\
z_3 &= \frac{1}{2}(\zeta_1\cos\theta-\zeta_2\sin\theta-\zeta_3\cos\theta-\zeta_4\sin\theta)\\
z_4 &= \frac{1}{2}\frac{Pa}{\nu 2B\cos\varphi}(\zeta_1\sin\theta+\zeta_2\cos\theta+\zeta_3\sin\theta-\zeta_4\cos\theta)
\end{aligned} \tag{4.6}$$

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